Separable Transformation

G is separable if

$$G(f) = A f B \quad \text{for some matrix } A, B.$$

Remark

$\vec{f}_i$: i-th col

$\vec{f}'_j$: j-th row

$$A f = A [\vec{f}_1 | \cdots | \vec{f}_N] = [A\vec{f}_1 | \cdots | A\vec{f}_N]$$

$\therefore$ Operation on Columns

$$f B = \begin{bmatrix} - & \vec{f}'_1 & - \\ - & \vec{f}'_2 & - \\ & \vdots & \\ - & \vec{f}'_N & - \end{bmatrix} B = \begin{bmatrix} - & \vec{f}'_1 B & - \\ & \vdots & \\ - & \vec{f}'_N B & - \end{bmatrix}$$

$\therefore$ Operation on rows

It is called Separable as it can be separated into operations on rows and columns.

Example

Suppose a Transformation Matrix:

$$H = \begin{bmatrix} 2 & 0 & 8 & 0 \\ 1 & 2 & 4 & 8 \\ 6 & 0 & 4 & 0 \\ 3 & 6 & 2 & 4 \end{bmatrix}$$

Block Matrix Multiplication

A_i, B_j are matrices

$$A = \begin{bmatrix} A_1 & A_3 \\ A_2 & A_4 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & B_3 \\ B_2 & B_4 \end{bmatrix}$$ are block matrices

$$AB = \begin{bmatrix} A_1 B_1 + A_3 B_2 & A_1 B_3 + A_3 B_4 \\ A_2 B_1 + A_4 B_2 & A_2 B_3 + A_4 B_4 \end{bmatrix}$$

Provided that all matrix multiplications and additions are make sense

$$g = Hf = \begin{bmatrix} 2 & 0 & 8 & 0 \\ 1 & 2 & 4 & 8 \\ 6 & 0 & 4 & 0 \\ 3 & 6 & 2 & 4 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \overbrace{\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}}^{=A} & 4 \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \\ 3 \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} & 2 \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \\ \begin{bmatrix} f_3 \\ f_4 \end{bmatrix} \end{bmatrix}$$

$$g = \begin{bmatrix} 1A \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + 4A \begin{bmatrix} f_3 \\ f_4 \end{bmatrix} \\ 3A \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + 2A \begin{bmatrix} f_3 \\ f_4 \end{bmatrix} \end{bmatrix}$$

reshape $\downarrow$

$$g = \begin{bmatrix} 1A \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + 4A \begin{bmatrix} f_3 \\ f_4 \end{bmatrix} & 3A \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + 2A \begin{bmatrix} f_3 \\ f_4 \end{bmatrix} \end{bmatrix}$$

$$= A \begin{bmatrix} f_1 & f_3 \\ f_2 & f_4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$$

$\therefore$ Separable, $O(f) = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} f \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$

Frobenius Norm

$$A \in \mathbb{R}^{N \times N}$$

$$\| A \|_F := \sqrt{\sum\sum_{1 \le i,j \le N} |a_{ij}|^2}$$

$\| A - B \|_F$ can be a measure of similarity between A and B.

e.g.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Find α s.t. $\| A - \alpha B \|_F$ is minimized.

Note minimizing $\| A - \alpha B \|_F$
$\Leftrightarrow$ minimizing $\| A - \alpha B \|_F^2$

$$\| A - \alpha B \|_F^2 = \sum\sum_{1 \le i,j \le 3} (a_{ij} - \alpha)^2$$

$$\frac{\partial}{\partial \alpha} \| A - \alpha B \|_F^2 = \sum\sum_{1 \le i,j \le 3} 2(\alpha - a_{ij})$$

$$\frac{\partial}{\partial \alpha} \| A - \alpha B \|_F^2 = 0$$

$$\Leftrightarrow \quad 3^2 \alpha = \sum\sum_{1 \le i,j \le 3} a_{ij} \Leftrightarrow \alpha = \frac{1}{9} \sum\sum_{1 \le i,j \le 3} a_{ij}$$

$\therefore$ α is the average of values of A.

SVD

$A \in \mathbb{R}^{M \times N}$

A can be decomposed to:

$$A = U \Sigma V^T$$

where $U \in \mathbb{R}^{M \times M}$, orthogonal

$\Sigma \in \mathbb{R}^{M \times N}$, non-negative diagonal, 0 elsewhere

$V \in \mathbb{R}^{N \times N}$, orthogonal

Example

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Note $A = U \Sigma V^T$

$$A A^T = (U \Sigma V^T)(V \Sigma^T U^T)$$
$$= U (\Sigma \Sigma^T) U^T$$
$$A^T A = (V \Sigma^T U^T)(U \Sigma V^T)$$
$$= V (\Sigma^T \Sigma) V^T$$

where $\Sigma\Sigma^T$, $\Sigma^T\Sigma$ are

diagonal square matrix.

So finding the diagonalization of

A^TA and AA^T gives us SVD.

To find eigenvectors and eigenvalues of AA^T:

$$AA^T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$0 = \det(AA^T - \lambda I) = (2-\lambda)^2 - 1$$

$$0 = \lambda^2 - 4\lambda + 3$$

$$\Leftrightarrow \quad 0 = (\lambda - 1)(\lambda - 3)$$

$$\Leftrightarrow \quad \lambda = 3 \text{ or } \lambda = 1.$$

$\lambda = 3$:

$$\begin{bmatrix} 2-3 & 1 \\ 1 & 2-3 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ eignvector $= a_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = 1$:

$$\begin{bmatrix} 2-1 & 1 \\ 1 & 2-1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ eigenvector $= a_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Normalizing, $U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

with $\Sigma = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$$A = U \Sigma V^T$$

$\vec{u}_i$: i-th col of U

$$U^T A = \Sigma V^T$$

$\vec{v}_j$: j-th of V

$$\begin{bmatrix} - \vec{u}_1^T - \\ - \vec{u}_2^T - \end{bmatrix} A = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{bmatrix} \begin{bmatrix} - \vec{v}_1^T - \\ - \vec{v}_2^T - \\ - \vec{v}_3^T - \end{bmatrix}$$

$$\begin{bmatrix} - (A^T \vec{u}_1)^T - \\ - (A^T \vec{u}_2)^T - \end{bmatrix} = \begin{bmatrix} - \sigma_1 \vec{v}_1^T - \\ - \sigma_2 \vec{v}_2^T - \end{bmatrix}$$

$$\therefore \vec{v}_1 = \frac{1}{\sigma_1} A^T \vec{u}_1, \quad \vec{v}_2 = \frac{1}{\sigma_2} A^T \vec{u}_2$$

$$\vec{v}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{6} \\ \sqrt{2/3} \\ 1/\sqrt{6} \end{bmatrix}$$

$$\vec{v}_2 = \frac{1}{1}\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

$\vec{v}_3$ can be calculated by cross product:

$$\vec{V}_3 = \vec{v}_1 \times \vec{v}_2$$

$$= \det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1/\sqrt{6} & \sqrt{2/3} & 1/\sqrt{6} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}$$

$\therefore \quad A = U \Sigma V^T$

where $U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

$$\Sigma = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$V = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \\ \sqrt{2/3} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & -1/\sqrt{3} \end{bmatrix}$$